1. On a Lower Boundary Condition for the Gravitational Force

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Introduction

In part one of this paper, the Universe is defined as a finite space that contains a finite number of gravitationally bound collections of matter. For clarity, one model of a static universe is defined to contain a single point-mass and compared to a model of a static universe containing two point-masses; these models are later generalized to a model containing N-number of point-masses for the purpose of examining constraints on the cosmological structure as it applies to a specific set of boundary conditions.

The point-masses are defined to be at the centers of galaxies: individual systems containing 1) a region in a state of continual negative gravitational contraction, and 2) a boundary where the gravitational force equals zero. The total gravitational energy contained within the zero boundary of all such systems is hypothesized to equal the total energy outside the boundary.

Finally, the hypothesis is examined using dimensional analysis.

A Universe of Point Masses

1.1 N = 1

Starting with the inverse square law of force, where the numerator expresses the product of a point-mass, a negligible reference-mass, and the Universal Gravitational Constant and thus determines the gravitational force for every distance,

$$f(r) = -\frac{K}{r^2}, \tag{1}$$

assume the existence of a non-zero positive constant.

$$f_N(r) = -\frac{K_N}{r^2} + B_N, \quad B_N = \frac{K_N}{b_N^2}, \tag{2}$$

Here B_N is a numeral not a function, and b_N represents the distance from the point-mass where the force is null, and beyond which it is positive, taking the form:

$$f_N(r) = -\frac{K_N}{r^2} + \frac{K_N}{b_N^2}$$

Examine the integral of the force, defined as the gravitational potential energy, while constraining the domain and range according to the following boundary conditions:

$$N\int_{a_N}^{b_N} f_N(r) \;=\; -N\int_{b_N}^{c_N} f_N(r), \qquad 0<a<b<c<\infty \tag{3}$$

$$f_1(a_1) \;=\; N f_N(a_N) \tag{4}$$

$$N(c_N - a_N) \;=\; \pi R \tag{5}$$

Where R is the radius of the universe.

The subscript, N, denotes the number of regions of equal mass in the universe, labeled the N-state. For example if N=1, equation (3) would represent a state where all the negative gravitational energy of the Universe constitutes the single point-mass, while all the positive gravitational energy constitutes the single region within which the gravitational force is positive and both distributed according to the inverse square law of force. Equation (5) would represent the total distance within which the function operates; if N=1, then the distance would be half the circumference of the Universe. See **Fig. 1**.

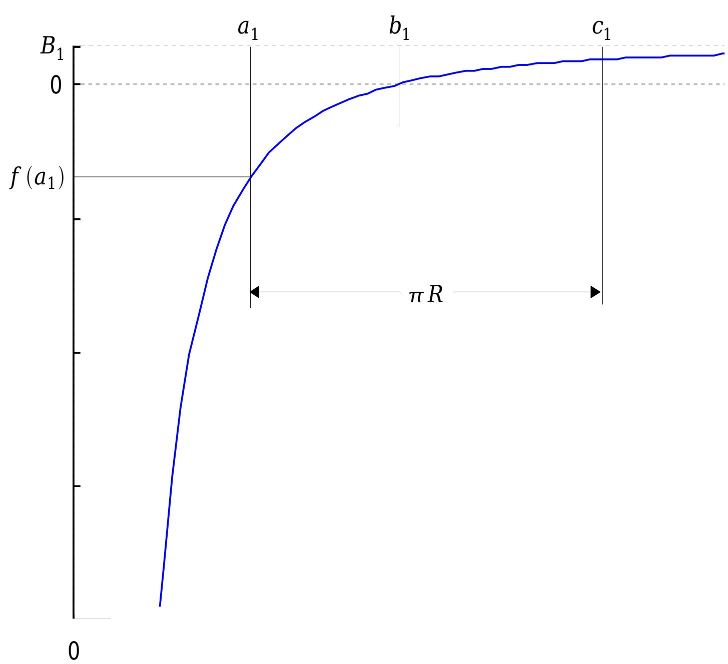

Fig.1 *Distribution of Gravitational Force in a N-Point-Mass Universe, where N=1, and $2(c_1 - a_1)$ = Circumference of the Universe.*

Values $\{a_N, b_N, c_N\}$ that satisfy the conditions of (3) relate accordingly. See Appendix A for proof. The subscripts are dropped for simplicity.

$$\{\, b^2 = a\,c \,\} \tag{6}$$

$$\left\{ \frac{a}{c} = \frac{(b+a)^2}{(c+b)^2} = \frac{b^2+a^2}{c^2+b^2} = \frac{(b-a)^2}{(c-b)^2} = \frac{b^2-a^2}{c^2-b^2} = \frac{b^2}{c^2} \right\} \tag{7}$$

$$\left\{ \frac{b+a}{c+b} = \frac{b-a}{c-b} = \frac{b}{c} \right\}. \tag{8}$$

The values $\{a, b, c\}$ are interpreted as follows:

$$\begin{aligned} a &= f^{-1}(maximum\,negative\,gravitational\ force) \\ b &= f^{-1}\,(gravitational\,force\,is\,null) \\ c &= f^{-1}\,(maximum\,positive\,gravitational\,potential). \end{aligned}$$

1.2 A Universe of Two Point Masses

While recalling that the subscript denotes the number of point-masses in the Universe, the relations between the values $\{a_{1,} b_{1,} c_1\}$ and $\{a_{2,} b_{2,} c_2\}$ are investigated with respect to the boundary conditions. To emphasize, an N=1 Universe, is a different universe than an N=2 Universe, though both may satisfy the conditions of containing an equal total gravitational energy, gravitational mass, and exist within a space of equal linear dimension.

For example, starting with equation (3), for a 2-mass Universe:

$$N \int_{a_N}^{b_N} f_N(r) \;=\; -N \int_{b_N}^{c_N} f_N(r), \quad 0<a<b<c<\infty \tag{3}$$

$$1\int_{a_1}^{b_1} f_1(r) \;=\; -1\int_{b_1}^{c_1} f_1(r) \;=\; 2\int_{a_2}^{b_2} f_2(r) \;=\; -2\int_{b_2}^{c_2} f_2(r), \quad 0<a_N<b_N<c_N<\infty \tag{9}$$

For N=2, equation (10) leads to the following relations. See Appendix A for proofs.

$$\left\{\frac{(b_1-a_1)^2}{a_1(b_1)^2} = \frac{(c_1-b_1)^2}{c_1(b_1)^2} = 2\alpha\left[\frac{(b_2-a_2)^2}{a_2(b_2)^2}\right] = 2\alpha\left[\frac{(c_2-b_2)^2}{c_2(b_2)^2}\right]\right\} \tag{10}$$

$$(b_1-a_1)=\sqrt{2}(b_2-a_2) \tag{11}$$

Fig. 2. compares the relative values $\{a_{2,} b_{2,} c_2\}$, $\{a_{1,} b_{1,} c_1\}$.

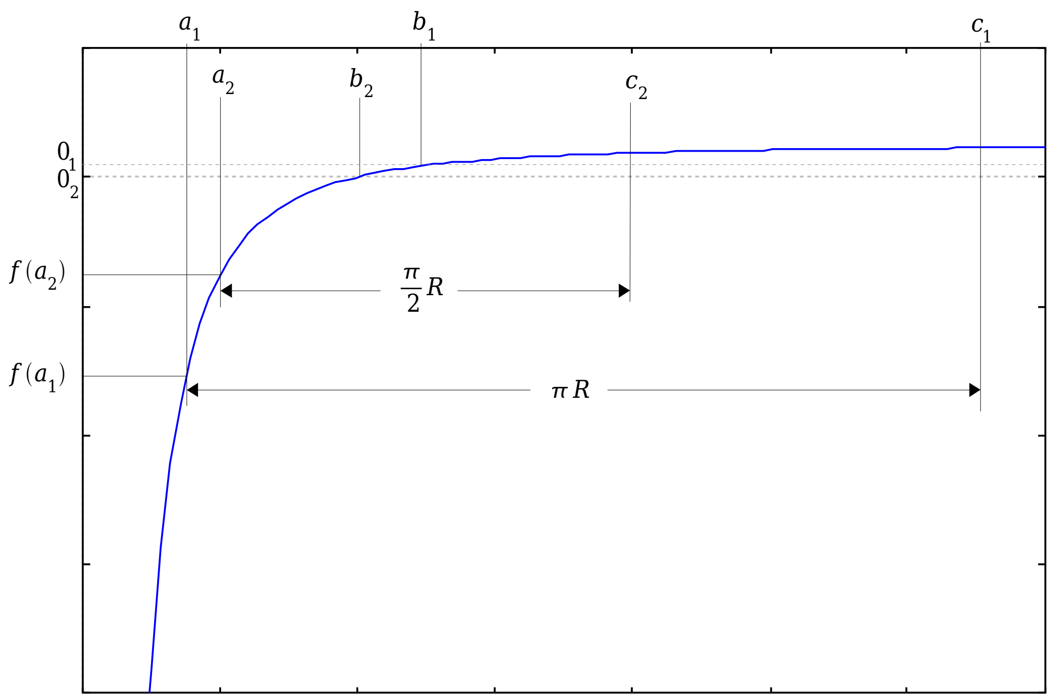

Fig. 2. *Values $\{a_N, b_N, c_N\}; N=1,2$ according to boundary conditions.*

1.3 N is Limited

Of course, the goal is to determine the state number N of our universe; later, it will be understood that the number N represents an integral number of average-sized galaxies (we imagine in the hundreds of billions) separated by regions of positive gravitational force.

The state number, N, is limited in this way: because $\frac{c_N}{a_N}$ must be greater than one (excluding any factor of proportionality), by (7), $\frac{(c_N - b_N)^2}{(b_N - a_N)^2}$ must also be greater than one: therefore, by the relations (4), and (A4.9), the state number, N, must be less than $\frac{(c_1 - a_1)^2}{4(b_1 - a_1)^2}$. Unsurprisingly, the number of average point masses, or say, the maximum number of collections of matter whose masses are equal (in this limited case), is constrained by the amount of space, and by the amount of mass. See Appendix for proof of the following:

$$N \leq \frac{(c_1 - a_1)^2}{4(b_1 - a_1)^2}. \tag{12}$$

Equation (12) states the larger is c_1, and the smaller is a_1, the larger N possible; and can be used as a galaxy correlation function. For example, assume a_1 is negligible in comparison to b_1 and c_1, then:

$$N \leq \frac{(c_1)^2}{4(b_1)^2}. \tag{13}$$

The second assumption, to be developed later, is that b_1 lies just outside the nuclear galactic bulge; so that equation (13) can be used to determine the circumference of the universe, given the average number of galaxies per unit distance along an average line of sight; or vice-versa.

1.4 Dimensional Analysis

Equation (13) states that the number of galaxies of average mass is approximately equal to or less than the radius of the universe squared divided by four times the square of the radius of the galaxy's region in which the gravitational field is negative. Restated in general terms:

$$N \leq \frac{c^2}{4b^2}. \tag{14}$$

The galactic core, sometimes referred to as the galactic nucleus, or bulge, is a central spherical region of a galaxy where the orbital motion of stars about the central axis is Keplerian. Outside this region, the motion of visible matter moves with constant velocity independent of it's distance from the galactic center, contradicting Kepler's Laws, and it is speculated by an overwhelming number of scientists that large amounts of unseen matter, coined Dark Matter, exists in a halo outside the galactic nucleus and determines the orbital motions of stars. This view is maintained so that, in theory, the sign of gravitational force remains only negative.

In this paper, *b* is defined as the boundary between negative and positive gravitational force; inside *b*, the sign of the gravitational field is negative and obeys the inverse square law of force so orbital motions are Keplerian. The boundary, *b*, is therefore the boundary containing the region of the galactic core; outside *b* motions are yet to be defined. A simple calculation of Equation 14, using nominally accepted values for *b*, and *c*, shows the results for N are reasonable:

b	***c***	***N***
5000 ly	10 billion ly	1 trillion
5000	15 billion	2.25 trillion
5000	20 billion	4 trillion
10000	10 billion	250 billion
10000	15 billion	560 billion
10000	20 billion	1 trillion

Appendix A

1 Equation (6)

$$N\int_{a_N}^{b_N} f_N(r) \; = \; -N\int_{b_N}^{c_N} f_N(r), \quad 0<a_N<b_N<c_N<\infty \tag{3}$$

If N=1

$$1\int_{a_1}^{b_1} f_1(r) \; = \; -1\int_{b_1}^{c_1} f_1(r), \quad 0<a_1<b_1<c_1<\infty. \tag{A1.1}$$

To prove equation (6), expand the Integrals:

$$\int_{a_1}^{b_1} \left(-\frac{k_1}{r^2}+\frac{k_1}{(b_1)^2}\right) dr \; = \; -\int_{b_1}^{c_1} \left(-\frac{k_1}{r^2}+\frac{k_1}{(b_1)^2}\right) dr. \tag{A1.2}$$

$$\left(\frac{k_1}{b_1}+\frac{k_1 b_1}{(b_1)^2}\right) - \left(\frac{k_1}{a_1}+\frac{k_1 a_1}{(b_1)^2}\right) = -\left[\left(\frac{k_1}{c_1}+\frac{k_1 c_1}{(b_1)^2}\right) - \left(\frac{k_1}{b_1}+\frac{k_1 b_1}{(b_1)^2}\right)\right]. \tag{A1.3}$$

The expressions with k_1 and b_1 cancel; divide out k_1 and multiply by -1:

$$\left(\frac{1}{a_1}+\frac{a_1}{(b_1)^2}\right) = \left(\frac{1}{c_1}+\frac{c_1}{(b_1)^2}\right). \tag{A1.4}$$

Add the expressions with like denominators:

$$\left(\frac{1}{a_1}-\frac{1}{c_1}\right) = \left(\frac{c_1}{(b_1)^2}-\frac{a_1}{(b_1)^2}\right). \tag{A1.5}$$

Simplify the left-hand side:

$$\left(\frac{a_1-c_1}{a_1 c_1}\right) = \left(\frac{a_1-c_1}{(b_1)^2}\right). \tag{A1.6}$$

Therefore,

$$(b_1)^2 \; = \; a_1 c_1. \tag{Q.E.D.}$$

2 Equations (7)

To Prove equation (7), start with (A1.5), add the terms on each side respectively:

$$\frac{(b_1)^2 + (a_1)^2}{a_1 (b_1)^2} = \frac{(c_1)^2 + (b_1)^2}{c_1 (b_1)^2} \tag{A2.1}$$

The term $(b_1)^2$ cancels; then, cross multiply:

$$\frac{(b_1)^2 + (a_1)^2}{(c_1)^2 + (b_1)^2} = \frac{a_1}{c_1} \qquad \text{(Q.E.D.)}$$

Multiply the right-hand side by $\frac{c_1}{c_1}$, recalling that $(b_1)^2 = a_1 c_1$:

$$\frac{(b_1)^2}{(c_1)^2} = \frac{a_1}{c_1} \qquad \text{(Q.E.D.)}$$

Starting with (A1.4), add $\frac{2 b_1}{(b_1)^2}$ to both sides:

$$\left(\frac{1}{a_1} + \frac{a_1}{(b_1)^2} + \frac{2 b_1}{(b_1)^2}\right) = \left(\frac{1}{c_1} + \frac{c_1}{(b_1)^2} + \frac{2 b_1}{(b_1)^2}\right). \tag{A2.2}$$

Add the terms on each side respectively:

$$\left(\frac{(b_1)^2}{a_1 (b_1)^2} + \frac{a_1 a_1}{a_1 (b_1)^2} + \frac{2 a_1 b_1}{a_1 (b_1)^2}\right) = \left(\frac{(b_1)^2}{c_1 (b_1)^2} + \frac{c_1 c_1}{c_1 (b_1)^2} + \frac{2 b_1 c_1}{c_1 (b_1)^2}\right). \tag{A2.3}$$

The terms $(b_1)^2$ cancel; simplify, then cross multiply:

$$\frac{(b_1 + a_1)^2}{(c_1 + b_1)^2} = \frac{a_1}{c_1}. \qquad \text{(Q.E.D.)}$$

Starting with (A2.1), subtract $\frac{2 b_1}{(b_1)^2}$ from both sides:

$$\frac{(b_1)^2 + (a_1)^2 - 2 a_1 b_1}{a_1 (b_1)^2} = \frac{(c_1)^2 + (b_1)^2 - 2 c_1 b_1}{c_1 (b_1)^2} \tag{A2.4}$$

The terms $(b_1)^2$ cancel; simplify, then cross multiply:

$$\frac{(b_1 - a_1)^2}{(c_1 - b_1)^2} = \frac{a_1}{c_1} \qquad \text{(Q.E.D.)}$$

Starting with (A.15), multiply the right-hand side by $\frac{(c_1 - a_1)}{(c_1 - a_1)}$:

$$\frac{(b_1 - a_1)^2}{(c_1 - b_1)^2} = \frac{a_1 (c_1 - a_1)}{c_1 (c_1 - a_1)}. \tag{A2.5}$$

Simplify, recalling that $(b_1)^2 = a_1 c_1$:

$$\frac{(b_1-a_1)^2}{(c_1-b_1)^2}=\frac{(b_1)^2-(a_1)^2}{(c_1)^2-(b_1)^2}. \qquad \text{(Q.E.D.)}$$

This completes the proofs.

3 Equation (10)

Starting with equation (9):

$$1\int_{a_1}^{b_1} f_1(r) \;=\; -1\int_{b_1}^{c_1} f_1(r) \;=\; 2\int_{a_2}^{b_2} f_2(r) \;=\; -2\int_{b_2}^{c_2} f_2(r), \quad 0<a_N<b_N<c_N<\infty$$

Expand the integrals:

$$\left(\frac{k_1}{b_1}+\frac{k_1 b_1}{(b_1)^2}\right) - \left(\frac{k_1}{a_1}+\frac{k_1 a_1}{(b_1)^2}\right) = -\left[\left(\frac{k_1}{c_1}+\frac{k_1 c_1}{(b_1)^2}\right) - \left(\frac{k_1}{b_1}+\frac{k_1 b_1}{(b_1)^2}\right)\right]$$

$$=2\left[\left(\frac{k_2}{b_2}+\frac{k_2 b_2}{(b_2)^2}\right) - \left(\frac{k_2}{a_2}+\frac{k_2 a_2}{(b_2)^2}\right)\right] = -2\left[\left(\frac{k_2}{c_2}+\frac{k_2 c_2}{(b_2)^2}\right) - \left(\frac{k_2}{b_2}+\frac{k_2 b_2}{(b_2)^2}\right)\right]. \qquad \text{(A3.1)}$$

The factor k_1 determines the magnitude of the gravitational force, and the total energy distribution for a single point-mass Universe; likewise, the factor k_2 determines the same for a 2-point-mass Universe. However, k_2 is in a Universe of a different state; the only assumption as to it's magnitude in relation to k_1 and to the constraints of equations (3), (4), and (5) is one of only proportion; so that: thom

$$k_2 \;=\; \alpha_2 k_1 \qquad \text{(A3.2)}$$

Rewriting the last half of equation (A3.1) with this substitution:

$$2\left[\left(\frac{\alpha_2 k_1}{b_2}+\frac{\alpha_2 k_1 b_2}{(b_2)^2}\right)-\left(\frac{\alpha_2 k_1}{a_2}+\frac{\alpha_2 k_1 a_1}{(b_2)} r^2\right)\right]=-2\left[\left(\frac{\alpha_2 k_1}{c_2}+\frac{\alpha_2 k_1 c_2}{(b_2)^2}\right)-\left(\frac{\alpha_2 k_1}{b_2}+\frac{\alpha_2 k_1 b_2}{(b_2)^2}\right)\right]. \qquad \text{(A3.3)}$$

Simplifying the whole of equation (A3.1) by factoring out K_1 , moving α_2 to the front:

$$\left(\frac{1}{b_1}+\frac{1}{b_1}\right) - \left(\frac{1}{a_1}+\frac{a_1}{(b_1)^2}\right) = -\left[\left(\frac{1}{c_1}+\frac{c_1}{(b_1)^2}\right) - \left(\frac{1}{b_1}+\frac{1}{b_1}\right)\right]$$

$$=2\alpha_2\left[\left(\frac{1}{b_2}+\frac{1}{b_2}\right) - \left(\frac{1}{a_2}+\frac{a_1}{(b_2)^2}\right)\right] = -2\alpha_2\left[\left(\frac{1}{c_2}+\frac{c_2}{(b_2)^2}\right) - \left(\frac{1}{b_2}+\frac{1}{b_2}\right)\right]. \qquad \text{(A3.4)}$$

Add the expressions with like denominators:

$$\frac{2}{b_1}-\left(\frac{1}{a_1}+\frac{a_1}{(b_1)^2}\right)=-\left[\left(\frac{1}{c_1}+\frac{c_1}{(b_1)^2}\right)-\frac{2}{b_1}\right]=2\alpha_2\left[\frac{2}{b_2}-\left(\frac{1}{a_2}+\frac{a_1}{(b_2)^2}\right)\right]=-2\alpha_2\left[\left(\frac{1}{c_2}+\frac{c_2}{(b_2)^2}\right)-\frac{2}{b_2}\right]. \qquad \text{(A3.5)}$$

Find common denominators:

$$\frac{2a_1b_1}{a_1(b_1)^2}-\left(\frac{(b_1)^2}{a_1(b_1)^2}+\frac{(a_1)^2}{a_1(b_1)^2}\right)=-\left[\left(\frac{(b_1)^2}{(b_1)^2c_1}+\frac{(c_1)^2}{(b_1)^2c_1}\right)-\frac{2b_1c_1}{c_1(b_1)^2}\right]$$

$$= 2\alpha_2\left[\frac{2a_2b_2}{a_2(b_2)^2}-\left(\frac{(b_2)^2}{a_2(b_2)^2}+\frac{(a_2)^2}{a_2(b_2)^2}\right)\right]=-2\alpha_2\left[\left(\frac{(b_2)^2}{(b_2)^2c_2}+\frac{(c_2)^2}{(b_2)^2c_2}\right)-\frac{2b_2c_2}{c_2(b_2)^2}\right] \qquad \text{(A3.6)}$$

Multiply by -1, add like terms (which are perfect squares), and simplify:

$$\frac{(b_1-a_1)^2}{a_1(b_1)^2} = \frac{(b_1-c_1)^2}{c_1(b_1)^2} = 2\alpha_2\left[\frac{(b_2-a_2)^2}{a_2(b_2)^2}\right] = 2\alpha_2\left[\frac{(b_2-c_2)^2}{c_2(b_2)^2}\right] \qquad \text{(A3.7)}$$

Because $(-X)^2=X^2$, the expressions may be written:

$$\frac{(b_1-a_1)^2}{a_1(b_1)^2} = \frac{(c_1-b_1)^2}{c_1(b_1)^2} = 2\alpha_2\left[\frac{(b_2-a_2)^2}{a_2(b_2)^2}\right] = 2\alpha_2\left[\frac{(c_2-b_2)^2}{c_2(b_2)^2}\right] \qquad \text{(Q.E.D.)}$$

4 Equation (11)

First, determine α, using equation (4):

$$N f_N(a_N) = N f_N(a_N) \qquad \text{(5)}$$

$$1 f_1(a_1) = 2 f_2(a_2) \qquad \text{(A4.1)}$$

$$\frac{-k_1}{(a_1)^2}+\frac{k_1}{(b_1)^2} = 2\left(\frac{-\alpha_2 k_1}{(a_2)^2}+\frac{\alpha_2 k_1}{(b_2)^2}\right) \qquad \text{(A4.2)}$$

Recall that $(b_N)^2=a_N c_N$, so that:

$$\frac{-k_1}{(a_1)^2}+\frac{k_1}{a_1c_1} = 2\alpha_2\left(\frac{-k_1}{(a_2)^2}+\frac{k_1}{a_2c_2}\right) \qquad \text{(A4.3)}$$

Factor out k_1 , multiply by -1:

$$\frac{1}{(a_1)^2}-\frac{1}{a_1 c_1} = 2\alpha_2\left(\frac{1}{(a_2)^2}-\frac{1}{a_2 c_2}\right) \quad \text{(A4.4)}$$

Add terms with like subscripts:

$$\frac{(c_1-a_1)}{(a_1)^2 c_1} = 2\alpha_2\left(\frac{(c_2-a_2)}{(a_2)^2 c_2}\right) \quad \text{(A4.5)}$$

Cross multiply, multiply inside parentheses by 2, and divide the factors in front by 2:

$$\frac{(a_2)^2 c_2}{(a_1)^2 c_1} = \alpha_2\left(\frac{2(c_2-a_2)}{(c_1-a_1)}\right) \quad \text{(A4.6)}$$

Recalling $(b_N)^2=a_n c_N$, rearrange the left-hand side. Recall by equation (5), that inside the right-hand side parentheses is equal to 1, so that:

$$\alpha_2 = \frac{a_2(b_2)^2}{a_1(b_1)^2}. \quad \text{(A4.7)}$$

Recall by equation (10):

$$\frac{(b_1-a_1)^2}{a_1(b_1)^2} = 2\alpha_2\left[\frac{(b_2-a_2)^2}{a_2(b_2)^2}\right] \quad \text{(A4.8)}$$

Make the substitution of equation (A4.7) into (A4.8) and simplify:

$$b_1-a_1 = \sqrt{2}(b_2-a_2) \quad \text{(Q.E.D.)}$$

The equation above is extended to N-states without proof.

$$b_1-a_1 = \sqrt{N}(b_N-a_N) \quad \text{(A4.9)}$$

5 Equation (12)

Starting with the relations in (7):

$$\frac{c_N}{a_N} = \left(\frac{(c_N-b_N)}{(b_N-a_N)}\right)^2 \geq 1 \quad \text{(A5.1)}$$

$$\frac{(c_N-b_N)}{(b_N-a_N)} \geq 1 \quad \text{(A5.2)}$$

Then, using equations (A4.9) and (4) in different form:

$$(c_N - a_N) = \frac{c_1 - a_1}{N} \qquad (4)$$

$$(b_N - a_N) = \frac{b_1 - a_1}{\sqrt{N}} \qquad (A4.9)$$

Using $(c_N - a_N) - (b_N - a_N) = (c_N - b_N)$, substitute (A4.9) and (4) into (A5.2):

$$\frac{\frac{c_1 - a_1}{N} - \frac{b_1 - a_1}{\sqrt{N}}}{\frac{b_1 - a_1}{\sqrt{N}}} \geq 1 \qquad (A5.3)$$

Rearrange terms, multiply last term on left-hand side by $\frac{\sqrt{N}}{\sqrt{N}}$:

$$\left(\frac{\sqrt{N}}{b_1 - a_1}\right)\left(\frac{c_1 - a_1}{N} - \frac{\sqrt{N}(b_1 - a_1)}{N}\right) \geq 1 \qquad (A5.4)$$

The square-root N cancels, multiply terms:

$$\frac{c_1 - a_1 - \sqrt{N}(b_1 - a_1)}{\sqrt{N}(b_1 - a_1)} \geq 1 \qquad (A5.5)$$

Split the numerator:

$$\frac{c_1 - a_1}{\sqrt{N}(b_1 - a_1)} - \frac{\sqrt{N}(b_1 - a_1)}{\sqrt{N}(b_1 - a_1)} \geq 1 \qquad (A5.6)$$

Add 1 to both sides:

$$\frac{c_1 - a_1}{\sqrt{N}(b_1 - a_1)} \geq 2 \qquad (A5.7)$$

Square both sides; then on both sides, divide by 4, and multiply by N:

$$\frac{(c_1 - a_1)^2}{4(b_1 - a_1)^2} \geq N \qquad (Q.E.D)$$